A proposal for non-commutative spectral synthesis

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Let G be a locally compact abelian group with dual group Γ. If I is a closed ideal of $L^1(G)$, then the zero set $Z(I)$ of I in Γ is defined as the set of common zeroes of all Fourier transform $\hat{f}$, for f in I. The classical (commutative) problem of spectral synthesis is then whether the closed set $Z(I)$ of Γ uniquely determines the ideal I. In this context, a closed set E of Γ, such that $E = Z(I)$ for a unique closed ideal I, is called an S-set. This problem has attracted considerable attention for roughly two decades, until it was answered by P. Malliavin in four 1959 papers: "Every closed subset of Γ is an S-set precisely when Γ is discrete."

When investigating the interplay between topological dynamics and associated Banach algebras, one naturally encounters a non-commutative version of the above situation. Namely, let $\Sigma = (X, \sigma)$ be a topological dynamical system, where X is a compact Hausdorff space and σ is a homeomorphism. Define the automorphism α of $C(X)$ by $\alpha(f)(x) = f(\sigma^{-1}x)$, and let the Banach *-algebra $\ell^1(\Sigma)$ consist of the functions $a : \mathbb{Z} \to C(X)$ with (finite) norm $\|a\| = \sum_n \|a(n)\|_\infty$. Product and *-operation are given as α-twisted convolution and involution, i.e., for $a : \mathbb{Z} \to C(X)$ and $b : \mathbb{Z} \to C(X)$, define

$$(ab)(n) = \sum_k a(k)\alpha^k(b(n-k)), \qquad a^\star(n) = \alpha^n(\overline{a(-n)}).$$

We let $C^*(\Sigma)$ denote the C^*-envelope of $\ell^1(\Sigma)$. Then $\ell^1(\Sigma)$ and $C^*(\Sigma)$ are commutative precisely when σ is the identity. If X consists of only one point, we recover familiar commutative algebras: In that case, $\ell^1(\Sigma) = \ell^1(\mathbb{Z})$ and $C^\star(\Sigma) = C(\mathbb{T})$, where the torus $\mathbb{T}$ is the dual group of $\mathbb{Z}$. The theory on the C^*-algebras $C^*(\Sigma)$ for general Σ is quite well developed, with also results for actions of $\mathbb{Z}^n$ being available, but the literature on $\ell^1(\Sigma)$ is currently limited to two papers by De Jeu, Svensson, and the lecturer. In this lecture, I propose, for general X, a non-commutative version of spectral synthesis for closed ideals of the involutive Banach *-algebra $\ell^1(\Sigma)$. New aspects arise here, because the dynamical system enters the picture. As it turns out, spectral synthesis holds in this context precisely when the dynamical system is free, i.e., precisely when there are no periodic points for the homeomorphism σ of X.